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UNSTEADY FORCED CONVECTIVE HEAT TRANSFER FROM A HOT FILM IN NON-REVERSING AND REVERSING SHEAR FLOW

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Abstract—The heat flux from a constant temperature hot film mounted flush with the surface of a plane insulating wall is examined numerically assuming that the thermal boundary layer theory is applicable everywhere and at all times. The flow field is described by (a) a pulsating or (b) a linearly decelerating two-dimensional shear flow. For high frequencies of the pulsation the calculated unsteady heat transfer departs strongly from the quasi-steady value. In reversing flow a considerable influence of the thermal wake can be seen. Computations of the time averaged heat flux show that an enhancement of heat transfer due to superimposed oscillations can only occur in reversing flow.

NOMENCLATURE

A_R,	amplitude response;
D,	thermal diffusivity [$m^2\ s^{-1}$];
L,	length of the hot film [m];
Nu,	Nusselt number, $\alpha L/\lambda$;
Nu^*,	modified Nusselt number, $Pe^{-1/3}Nu$;
Pe,	Péclet number, $S_{ref}L^2/D$;
Pr,	Prandtl number, ν/D;
R,	radius of the tube [m];
Re_{2R},	Reynolds number, $U_m 2R/\nu$;
s, S,	(dimensionless) velocity gradient at the wall [s^{-1}];
$\dot{s}, \dot{S}$,	(dimensionless) change of the velocity gradient at the wall [s^{-2}];
t,	time [s];
T,	temperature [K];
T_0,	free stream temperature [K];
T_1,	hot film temperature [K];
U_m,	mean velocity of laminar tube flow [$m\ s^{-1}$];
U_0,	free stream velocity [$m\ s^{-1}$];
x, X,	(dimensionless) coordinate parallel to the surface [m];
y, Y,	(dimensionless) coordinate normal to the surface [m].

Greek symbols

α,	heat transfer coefficient [$W\ m^{-2}\ K^{-1}$];
Γ,	Gamma function;
δ_T,	thickness of the thermal boundary layer [m];
$\Delta x, \Delta y, \Delta\tau$,	steps in the appropriate coordinate directions;
η,	similarity variable, $y(s/9x)^{1/3}$;
θ,	dimensionless temperature;
λ,	thermal conductivity [$W\ m^{-1}\ K^{-1}$];
μ,	dynamic viscosity [$kg\ m^{-1}\ s^{-1}$];
ν,	kinematic viscosity [$m^2 s^{-1}$];
τ,	dimensionless time;
τ_w,	wall shear stress [$N\ m^{-2}$];
ϕ,	phase difference [degrees];
ω, Ω,	(dimensionless) angular frequency [s^{-1}].

Subscripts

i, j, k,	index for the coordinate directions x, y, τ, respectively;
in,	instantaneous;
max,	maximum;
min,	minimum;
n,	edge of boundary layer;
q,	quasi-steady;
ref,	reference value;
s,	steady;
u,	unsteady.

Superscripts

$\hat{}$,	amplitude;
$\bar{}$,	time averaged.

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1. INTRODUCTION

IN TWO-DIMENSIONAL steady incompressible flow the convective heat transfer from a small isothermal heated film on an insulated wall is (under ideal conditions) found to be approximately proportional to the one-third power of the local wall shear stress, τ_w. This relationship was first derived by Lévêque [1] and was used by Ludwieg [2] for the development of a hot-film probe for wall shear stress measurements. The one-third power law holds as long as the heat conduction in the flow direction is negligible in the thermal boundary layer that develops over the film, requiring the Péclet number, $Pe = \tau_w L^2/\mu D$, to be large and the thermal end effects to be small. Furthermore, the thermal boundary layer has to be confined to the region next to the wall where the velocity field can be approximated by a linear shear profile.

For given fluid properties the local wall shear in an incompressible steady boundary layer depends only on the shape of the boundary and the free stream velocity, U_0. (For a flat plate boundary layer Blasius' solution gives $\tau_w \propto U_0^{3/2}$.) Therefore a hot-film probe inserted into the stream can also be operated as an anemometer.

In unsteady flow the direct relationships between velocity, wall shear, and heat transfer are distorted by inertial effects. Measurements of unsteady wall shear stress and velocity distributions with hot-film probes rely on the assumption that this distortion is small. Thus, it is assumed that the unsteady heat transfer from the hot-film departs only a little from its quasi-steady value.

The aim of this paper is to examine theoretically how good the quasi-steady approximation is and to determine when it breaks down. It is, however, beyond the scope of this work to solve the full, coupled flow and heat transfer problem even for a particular case. So we restrict ourselves to the response of a hot film to a given shear variation. This enables us to obtain general results about the importance of the thermal inertia of the fluid in unsteady hot-film measurements. In addition, we will investigate the response of the probe to shear reversal. The heat output (which is always a positive quantity) cannot show the change of the flow direction. Furthermore, the thermal wake leads to different conditions in the oncoming (i.e. reversed) flow: the temperature of the wake is higher than that of the surrounding fluid so the overheat ratio is effectively reduced and the heat transfer falls below its quasi-steady value.

Much work has been done on the response of heat and mass transfer to unsteady flow. Most theoretical papers are, however, restricted to non-reversing flow. Theoretical investigations of heat transfer problems in reversing flow are very rare although this case is of particular importance because of the early wall shear reversal in unsteady flow.

One of the first papers on unsteady heat transfer from an isothermal surface is by Lighthill [3]. He studied the changes in the heat flux from a flat plate due to small fluctuations in the free stream velocity for high and low frequencies of the pulsation by means of a perturbation method. Similarly, Fagela-Alabastro and Hellums [4] investigated the concentration boundary layer in pulsatile pipe flow and obtained solutions for the local mass flux. Recently Lueck [5] extended Lighthill's theory and applied his results to hot-film probes commonly used for oceanographic measurements. The characteristic quantities for the quality of an unsteady signal, the amplitude and the phase response, were studied by Patel *et al.* [6] and Mizushina *et al.* [7], both theoretically and experimentally with mass transfer probes. Fortuna and Hanratty [8] calculated correction coefficients for the distorted hot-film signal in fluctuating (turbulent) flow. Using a von Karman–Pohlhausen method, Kurz [9] calculated the heat transfer from a hot film in unsteady flow in order to test the applicability of steady calibrations for unsteady measurements without flow reversal. Pedley [10] gives an asymptotic expansion solution for the heat transfer of a hot film in pulsatile flow. He extended this work to give a crude theory for flow with reversal [11] and used this method in connection with an unsteady boundary layer analysis [12] for a comparison with the unsteady hot-film calibration experiments of Clark [13] and Seed and Wood [14]. Good agreement was found except for a phase lag between the approximate theory and the experiments which could not be explained satisfactorily. As there also exists a phase shift between Pedley's solution and the present calculations, the phase lag might be due to the strong simplifications in Pedley's theory.

The results presented in this paper are expected to be applicable to hot film or electrochemical probes used for shear stress measurements. It is hoped that they provide enough information for the experimentalist to estimate the accuracy of his measurements. The results might also be of interest to the chemical engineer concerned with heat/mass transfer in unsteady flow.

2. STATEMENT OF THE PROBLEM

The hot film is modelled as a small heated strip of length L embedded in a plane insulated boundary. The film temperature, T_1, is taken to be constant. A fluid with constant thermal properties and initial temperature T_0 passes the boundary with a time-dependent linear shear flow and is heated by the film (Fig. 1). For this problem the energy equation can be written as

$$\frac{\partial T}{\partial t} + Y S(t) \frac{\partial T}{\partial X} = D\left[\frac{\partial^2 T}{\partial Y^2} + \frac{\partial^2 T}{\partial X^2}\right] \tag{1}$$

where T is the temperature and $S(t)$ the time dependent velocity gradient. We introduce the following dimensionless variables [of $O(1)$]

$$x = \frac{X}{L}, \quad y = \frac{Y}{L} Pe^{1/3},$$

$$S = \frac{S}{S_{ref}}, \quad \theta = \frac{T - T_0}{T_1 - T_0}$$

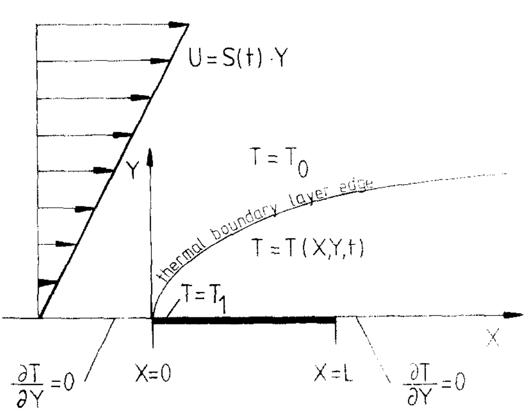

FIG. 1. Sketch of the idealized hot-film and flow field.

and the dimensionless time

$$\tau = \left[\frac{t}{(L^2/D)}\right] Pe^{2/3}$$

where the Péclet number is defined as

$$Pe = S_{ref}L^2/D. \quad (2)$$

S_{ref} is a reference value for the wall shear which has to be chosen according to the given shear function. The resulting dimensionless temperature equation is

$$\theta_\tau + ys(\tau)\theta_x = \theta_{yy} + Pe^{-2/3}\theta_{xx}. \quad (3)$$

Hot-film probes are commonly used in the high Péclet number range. Therefore, we make a boundary layer approximation and neglect the term for the heat conduction in the flow direction. Thus we obtain the equation for a 2-dim. unsteady thermal boundary layer,

$$\theta_\tau + ys(\tau)\theta_x = \theta_{yy}, \quad (4a)$$

which has to be solved subject to the following boundary conditions:

$$\begin{aligned} \theta(x, y = 0, \tau) &= 1 \quad \text{for} \quad 0 \leqslant x \leqslant 1, \\ \theta_y(x, y = 0, \tau) &= 0 \quad \text{for} \quad 0 < x, x > 1, \quad (4b) \\ \theta_x(x, y, \tau) &= 0 \quad \text{for} \quad y \to \infty, x \to -\infty. \end{aligned}$$

Various initial conditions may be appropriate in practice; for convenience we shall in general choose as the initial condition the solution of the steady state equation

$$ys(\tau = 0)\theta_x = \theta_{yy} \quad (4c)$$

subject to the same boundary conditions.

For the isolated region $0 < x < 1$ the solution of this equation is given by the well-known thermal boundary layer solution of Lévêque [1]

$$\theta = 1 - \frac{1}{\Gamma(4/3)} \int_0^\eta e^{-\eta'^3} \, d\eta' \quad (5)$$

with $\eta = y(s/9x)^{1/3}$ a similarity variable and $1/\Gamma(4/3) = 1.120$.

The steady wake for $x \gg 1$ is given by another similarity solution derived by Ling [15]

$$\theta = \frac{3Nu_s^*}{9^{2/3}\Gamma(2/3)} x^{-2/3} \, e^{-\eta^3} \quad (6)$$

where η is the same variable as above, $Nu_s^* = 0.8075$ as described below, and the numerical value of the gamma function is 1.3541.

The overall heat flux is described by the Nusselt number

$$Nu = \frac{\alpha L}{\lambda} = \int_0^L -\frac{\partial T}{\partial Y}\bigg|_{Y=0} \frac{dX}{(T - T_0)}$$

$$= Pe^{1/3} \int_0^1 -\theta_y|_{y=0} \, dx \quad (7a)$$

This expresses the well-known fact, that the Nusselt number for the Lévêque problem is proportional to the one-third power of the Péclet number. Because the Péclet number does not occur explicitly in equation (4) the heat transfer results are more conveniently presented in terms of a modified Nusselt number. This is independent of Pe and can be defined as

$$Nu^* = Pe^{-1/3}Nu = \int_0^1 -\theta_y|_{y=0} \, dx. \quad (7b)$$

Lévêque's solution for steady flow gives

$$Nu_s^* = 0.8075. \quad (7c)$$

We seek to calculate Nu^* as a function of τ for various shear functions $s(\tau)$.

For a given shear variation, $s(\tau)$, the temperature equations 4(a)–4(c) can be solved by numerical methods. The finite-difference technique used is described in Section 4.

3. DISCUSSION OF ASSUMPTIONS

3.1. *The thermal boundary layer approximation*

By neglecting the axial heat conduction over the whole length of the film we have excluded the possibility of thermal end effects. Clearly, the boundary layer approximation breaks down at the leading and trailing edges owing to the discontinuities in the temperature field: θ_{xx} approaches infinity at the ends and the term $Pe^{-2/3}\theta_{xx}$ in equation (3) cannot be neglected. However, Ling's [15] numerical solution of the full temperature equation for steady shear flow over a finite hot-film suggests that the growth of the overall heat transfer owing to axial conduction is less than 1% for $Pe \geqslant 500$. Springer and Pedley [16] and Springer [17] obtained analytical solutions for the leading and trailing edges of a long hot film using the Wiener–Hopf technique. Their results, too, show that for high Péclet numbers the influence of the ends on the overall heat flux from the film is negligibly small. Ackerberg *et al.* [18] performed steady mass flux experiments over a large range of the Péclet number. For high Pe their data agrees well with Lévêque's solution. As we want to determine the response (i.e. the overall heat transfer) of a hot film in an unsteady shear flow we will not lose very much accuracy by using equation (4) instead of the more complicated equation (3), so long as the instantaneous Péclet number, $Pe_{in}(t) = S(t)L^2/D$, is large.

However, if the unsteady shear approaches zero or passes through zero during reversal we again have to consider the importance of axial heat conduction. For very small shear rates the convective term cannot balance the vertical heat conduction. Thus, heat conduction in the flow direction must be important unless the thermal inertia term θ_τ is large enough to balance the vertical diffusion term.

If the inertia term is small the heat transfer proceeds in a quasi-steady fashion. Hence, an unsteady analysis of the behaviour of the hot-film probe is not necessary (as long as the shear does not reverse) and the steady theory of Ackerberg *et al.* [18] could be applied in the small Péclet number range. Furthermore, departures from boundary layer theory are also present when a probe is calibrated in steady flow. Thus, they are implemented in the calibration curve and the wall shear can be accurately predicted if this curve is used to relate the wall shear to the measured heat flux.

If thermal inertia is important, however, and it is this case in which we are mainly interested, we may assume that axial diffusion of heat is still negligible and that vertical diffusion dominates, being balanced by thermal inertia. This concept will now be discussed for the problem under consideration. During the deceleration of the shear the thickness of the thermal boundary layer is always less than one would expect from a quasi-steady theory. This is due to the fact that fluid particles have to be heated up as the boundary layer thickness increases. In other words the thermal inertia term is always positive (as is the term θ_{yy}) and reduces the speed of the diffusion process. If s approaches zero this means that the temperature field still resembles that of a faster flow with a thin boundary layer where the heat conduction in the flow direction is comparatively small. However, the balance in the temperature equation is no longer between diffusion and convection but between diffusion and thermal inertia. If the acceleration phase of the flow (or in the case of shear reversal the fast backflow) then takes over and no quasi-steady state with small shear rate can be established, "normal" boundary layer theory will again account for the neglect of the term $Pe^{-2/3}\theta_{xx}$. Consequently, we are justified in using equation (4) throughout our heat flux calculations provided the flow is so unsteady that θ_τ is large during the periods of low shear. We shall have to check the validity of this assumption for the particular functions $s(\tau)$ chosen in the next subsection.

In order to render the 2-dim. thermal boundary layer theory applicable to pulsatile pipe flow it has to be ensured that the boundary layer is very much thinner than the pipe radius R. The same condition must be imposed to permit the linearization of the flow field in the near wall region. For steady flow this condition can be checked. The linearization of the velocity profile is accurate to 1% as long as Y remains smaller than $0.020R$. Thus, the condition

$$\frac{\delta_T}{R} \leqslant 0.020 \tag{8}$$

must be imposed. Assuming the edge of the thermal boundary layer at $\theta = 0.99$ we have at the end of the hot film

$$\frac{\delta_T}{L} = 2.9Pe^{-1/3}. \tag{9}$$

With the friction law for laminar pipe flow

$$\frac{\nu S}{U_m^2} = \frac{8}{Re_{2R}} \tag{10}$$

we finally obtain for the condition (8)

$$2.3\left(\frac{L}{R}\right)^{1/3}(Re_{2R}Pr)^{-1/3} \leqslant 0.020. \tag{11}$$

This is valid for steady flow and gives only a very rough estimate for the conditions in unsteady flow where the curvature of the velocity profile is much more pronounced. Furthermore, there is a growth of the boundary layer in the wake which was not included in equation (11). Thus, the restrictions that must be imposed on δ_T/R will have to be stronger than equations (8) and (11), respectively. A full discussion of the above problem for hot-film probes mounted on a flat plate and inserted into a stream can be found in ref. [10].

3.2. *The shear function*

So far we have not given our choice of the form of the shear function. Because we want to investigate the response of a hot film to fluctuating flow we take it to be

$$S = S_{ref} + \hat{S}\cos\Omega t. \tag{12a}$$

This represents, for example, the wall shear in a pulsating pipe flow due to a pulsating pressure gradient [19]. In dimensionless terms we have

$$s = 1 + \hat{s}\cos\omega\tau \tag{12b}$$

where

$$\omega = \Omega\frac{L^2}{D}Pe^{-2/3}.$$

(With this definition of ω the inertia term θ_τ is of order ω and for $\omega \ll 1$ we have quasi-steady conditions.) As soon as the initial transients (caused by the initial conditions) are damped out we expect a periodic solution in time.

In order to show the influence of the thermal wake on the heat transfer other choices of the shear function are also considered:

(a) Linearly decelerating wall shear,

$$S = -\dot{S}t, \tag{13}$$

where the shear deceleration $\dot{S} > 0$ is kept constant. A sensible choice for the reference shear is here obviously the product of the shear deceleration $\dot{S}$ and the time scale of the system $(L^2/D)Pe^{-2/3}$ which leads to

$$S_{ref} = (\dot{S}^3L^2/D)^{1/5}. \tag{14}$$

Having this value for the reference shear we can define an unsteady Péclet number

$$Pe_u = \dot{S}^{3/5}(L^2/D)^{6/5} \tag{15}$$

and the temperature equation is

$$\theta_\tau - \tau y\theta_x = \theta_{yy}. \tag{16}$$

(b) A steady shear flow is decelerated through reversal until it reaches the same shear rate in the opposite direction and is then held constant again. For this case the reference shear value is taken to be that of the maximum shear.

It is now possible to verify the neglect of axial diffusion at times of low shear. For this purpose it is advantageous to linearize the shear function in the regime of shear reversal. Thus, the unsteady Péclet number Pe_u and equation (16) could be used to describe the temperature field. For large Pe_u the flow field is so unsteady that the time interval during which the shear rate is too low for the application of "normal" boundary layer theory is so small that significant changes in the temperature field can hardly occur. The axial heat conduction can therefore still be neglected if Pe_u as well as Pe is large.

In pulsating reversing flow the relationship between the unsteady and the reference Péclet number [equations (14) and (3), respectively] can be evaluated to give

$$Pe_u = \left(\frac{\dot{S}(S=0)}{S_{ref}^2}\right)^{3/5} Pe^{6/5} = [\omega(\hat{s}^2-1)^{1/2}]^{3/5} Pe. \quad (17)$$

4. NUMERICAL METHOD

The partial differential equation (4a) and the conditions (4b) and (4c) represent a combined initial and boundary value problem which can be solved by a finite-difference technique. Because the finite-difference technique is based upon a Taylor series expansion of the unknown variable (requiring continuous functions and derivatives) the singularities in the boundary conditions at $x = 0$ and $x = 1$ can cause a local instability giving an inaccurate solution near the edges of the hot film. This problem was resolved by using a local mesh refinement for the x-step. As the instability damps out very quickly after a few steps owing to "numerical viscosity" the inaccuracy of the solution in the neighbourhood of the singular points has negligible influence on the prediction of the overall heat transfer.

The initial steady state solution was programmed with a Crank–Nicholson scheme and a comparison of the solution with Lévêque's solution suggests that the error in Nu^* is always smaller than 0.1%.

For the unsteady temperature field calculation a Crank–Nicholson type finite-difference scheme, introduced by Krause *et al.* [20] was used as long as s was positive [Fig. 2(a)]. The discretization of the differential equation (4a) is then given by

$$[\theta_\tau]_{i-1/2,j,k-1/2} + \bar{s}y_j[\theta_x]_{i-1/2,j,k-1/2} = [\theta_{yy}]_{i-1/2,j,k-1/2} \quad (18)$$

where

$$[\theta_\tau]_{i-1/2,j,k-1/2} = \frac{1}{2\Delta\tau}[\theta_{i,j,k}-\theta_{i,j,k-1}+\theta_{i-1,j,k}-\theta_{i-1,j,k-1}]$$

$$[\theta_x]_{i-1/2,j,k-1/2} = \frac{1}{2\Delta x}[\theta_{i,j,k}-\theta_{i-1,j,k}+\theta_{i,j,k-1}-\theta_{i-1,j,k-1}]$$

$$[\theta_{yy}]_{i-1/2,j,k-1/2} = \frac{1}{2\Delta y^2}[\theta_{i,j+1,k}-2\theta_{i,j,k}+\theta_{i,j-1,k} + \theta_{i-1,j+1,k-1}-2\theta_{i-1,j,k-1}+\theta_{i-1,j-1,k-1}]$$

and

$$\bar{s} = \int_{\tau_{k-1}}^{\tau_k} s(\tau)\,d\tau \approx s_{k-1/2}.$$

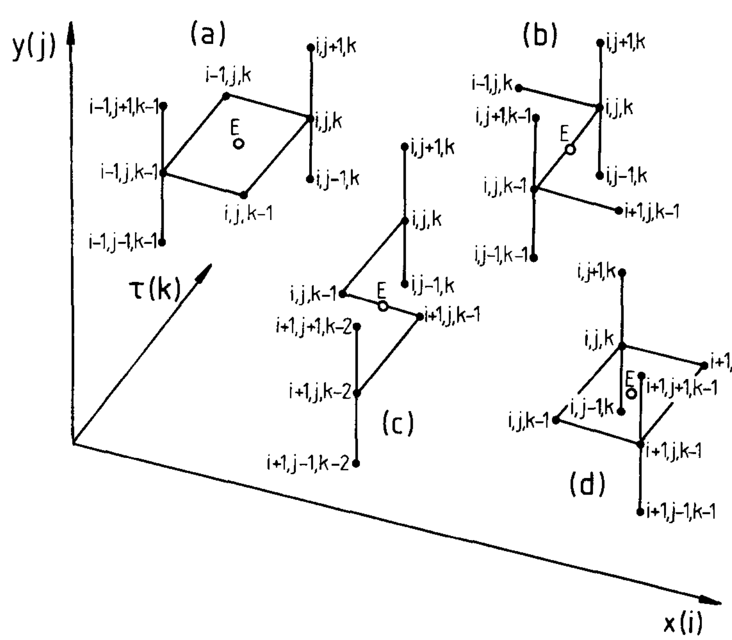

FIG. 2. Finite-difference schemes (E = expansion point).

The discretization error is of second order in Δx, Δy and $\Delta\tau$. For numerical stability it must be ensured that

$$\bar{s}y\frac{\Delta\tau}{\Delta x} \geqslant 0$$

which for positive $\bar{s}$ is always satisfied.

The discretization of the differential equation leads to a set of simultaneous linear equations for the unknowns $\theta_{i,j,k}$, $j = 1\ldots n$ (n denotes a point on the edge of the thermal boundary layer) at each position x_i, τ_k. The set will have tridiagonal matrix form and can be solved recursively making use of the boundary conditions and of previously calculated values of θ. The local heat transfer was calculated from the temperature field by an averaged 3- and 4-point formula, as it gave the most accurate results in a comparison with Lévêque's solution in steady flow,

$$[\theta_y]_{i,j=1,k} = \frac{1}{6\Delta y}[\theta_{i,j=4,k} - 6\theta_{i,j=3,k} + 15\theta_{i,j=2,k} - 10\theta_{i,j=1,k}]. \quad (19)$$

The same formula was used to approximate the insulated wall boundary condition. The overall heat flux was determined from the local heat transfer values by a simple trapezoidal rule integration.

The choice of a numerical scheme for the reversed shear ($s < 0$) is not as easy as that for positive shear. One possibility is the commonly used zig-zag scheme of Krause *et al.* [20] [Fig. 2(b)]. But its stability condition

$$\bar{s}y\frac{\Delta\tau}{\Delta x} \geqslant -1$$

imposes an unrealistic restriction on the stepsize $\Delta\tau$ because Δx has to be very small at the edges of the film (see above). Therefore the following procedure was adopted. Far downstream, in the wake, a new "boundary condition" was predicted using a newly developed 3-level scheme [Fig. 2(c)]. The discretization of the differential equation now gives

$$[\theta_\tau]_{i+1/2,j,k-1} + \bar{s}y_j[\theta_x]_{i+1/2,j,k-1} = [\theta_{yy}]_{i+1/2,j,k-1} \quad (20)$$

where

$$[\theta_\tau]_{i+1/2,j,k-1} = \frac{1}{2\Delta\tau}[\theta_{i,j,k} - \theta_{i,j,k-1} + \theta_{i+1,j,k-1} - \theta_{i+1,j,k-2}],$$

$$[\theta_x]_{i+1/2,j,k-1} = \frac{1}{\Delta x}[\theta_{i+1,j,k-1} - \theta_{i,j,k-1}],$$

$$[\theta_{yy}]_{i+1/2,j,k-1} = \frac{1}{2\Delta y^2}[\theta_{i,j+1,k} - 2\theta_{i,j,k} + \theta_{i,j-1,k} + \theta_{i+1,j+1,k-2} - 2\theta_{i+1,j,k-2} + \theta_{i+1,j-1,k-2}]$$

and

$$\bar{s} = \int_{\tau_{k-2}}^{\tau_k} s(\tau)\,d\tau \approx s_{k-1}.$$

The discretization error is again of second order and as stability condition we obtain from a von Neumann stability analysis

$$-1 \leqslant \bar{s}y\frac{\Delta\tau}{\Delta x} \leqslant 0.$$

[Scheme (b) could not be used because it has in this application four unknown points at τ_k.] Apparently, the restriction on the step $\Delta\tau$ seems to be the same as for scheme (b). However, the new scheme is only used to compute a new "boundary condition" in the former wake where a larger Δx can be chosen. The scheme (c) needs additional points at a previous time level $k-2$. But as these points are only required at the end of the computation field the increase in computer storage is only small.

The other temperature values can now easily be computed with a backwards facing (turned) scheme (a), progressing in the negative x-direction [Fig. 2(d)]. The question still to be answered is how far downstream in the wake it is necessary to go in order to assess its effect on heat transfer in reversed flow. This is discussed next.

4.1. *Zones of dependence*

The stability restrictions of the chosen finite difference approximations can easily be explained physically if we adopt Wang's [21] concept of characteristics and subcharacteristics.

The characteristics of equation (4a) are given by all lines normal to the surface; the speed of a temperature disturbance in their direction is infinite. By using an implicit formulation for the y-direction and solving the difference equation for all points y_j simultaneously the physical condition of infinite speed of diffusion can be adequately modelled. Therefore, no stability restriction based upon the Δy stepsize can occur.

The subcharacteristics of equation (4a) given by

$$sy\frac{d\tau}{dx} = -1 \quad (21)$$

(which describes the projection of the path of a fluid particle at fixed height y onto the x–τ plane) can, however, impose a restriction on the stepsize $\Delta\tau$. Writing equation (21) in difference form we have the condition for marginal stability of scheme (c)

$$\bar{s}y\frac{\Delta\tau}{\Delta x} = -1. \quad (22)$$

In the case of negative shear this means that a fluid particle at fixed height y_j which was previously (at τ_{k-1}) at the position $x_{i+1} = x_i + \Delta x$ is convected backwards to the position x_i in the time interval $\Delta\tau$ (Fig. 3). However, if

$$\bar{s}y\frac{\Delta\tau}{\Delta x} < -1$$

the particle will have passed the position x_i and will be at some place $x_i - \Delta x^*$. Thus, the fluid particle of x_i is the one that was previously at $x_{i+1} + \Delta x^*$. Now the temperature of the particle clearly depends on its

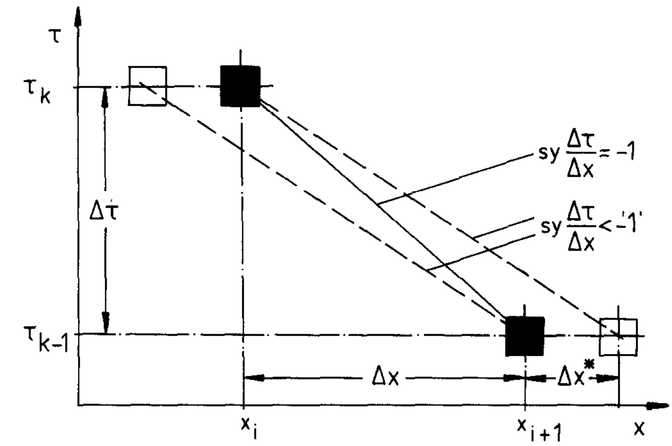

FIG. 3. View of the path of a fluid particle in the x–τ plane.

previous temperature, but this cannot be determined from the information provided within the numerical scheme because the particle's previous position lies outside the covered area. The prediction of $\theta_{i,j,k}$ must therefore be inaccurate and the numerical solution becomes unstable.

The concept of subcharacteristics can of course also be applied to the finite difference schemes (a) and (d). But there we always have a known upstream point giving the necessary information for calculating the new temperatures. Thus, these schemes are stable as long as the flow comes from the "upstream end" of the scheme.

The limiting subcharacteristics for the zone of dependence are

$$x_1 - x_0 = \int_{\tau_0}^{\tau_1} s y_{\max}\, d\tau \tag{23a}$$

and

$$x_1 - x_0 = 0. \tag{23b}$$

To be able to calculate the heat transfer we have to know the initial temperatures of all particles that will be convected over the film. If $x_1 = 1$ denotes the end of the heat transfer region, and a periodic solution (in the case of pulsating flow) can be expected after about 2 backflow periods, the integration of equation (23a) over all times of negative shear yields the point x_0 up to which the initial solution has to be calculated. Similarly, with $x_1 = 0$ and one interval of negative shear, we can determine a point x_0 in front of the hot-film up to which we must calculate in order to include all heated particles that will pass the film again during the following period of positive shear. During periods of backflow the computation field will decrease by one step Δx for each step $\Delta\tau$ and cannot be increased during periods of $s > 0$ because of the limiting subcharacteristic (23b).

The numerical procedure was tested in a simple heat transfer problem in reversing flow that was solved analytically by Pedley [21]. The numerical computation was accurate to within three significant figures.

5. PRESENTATION AND DISCUSSION OF RESULTS

The time dependent heat transfer for the different shear functions is presented in terms of the modified Nusselt number Nu^* in Figs. 4–9. The quality of the hot-film signal is judged by comparing the unsteady with the ideal, quasi-steady heat transfer, Nu_q^*.

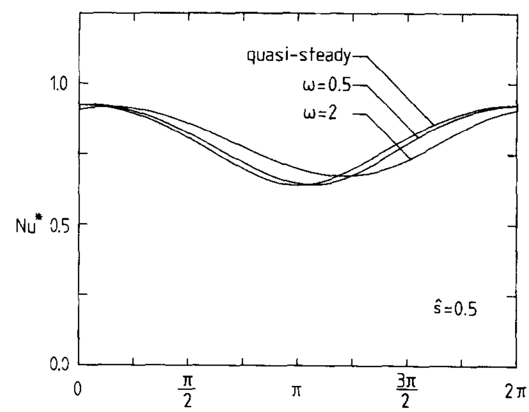

FIG. 4(a). Response of the hot film in non-reversing flow, $\hat{s} = 0.5$.

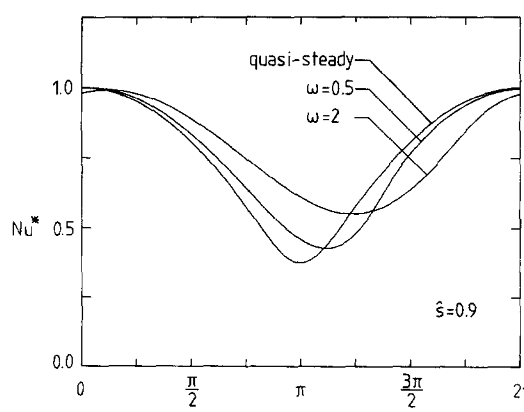

FIG. 4(b). Response of the hot film in non-reversing flow, $\hat{s} = 0.9$.

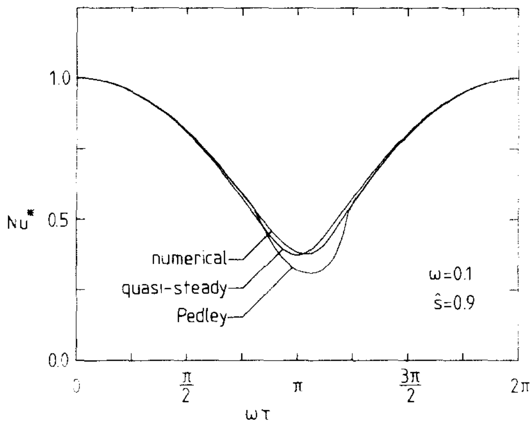

FIG. 5(a). Comparison with Pedley's solution [11] in non-reversing pulsatile flow ($\hat{s} = 0.9$, $\omega = 0.1$).

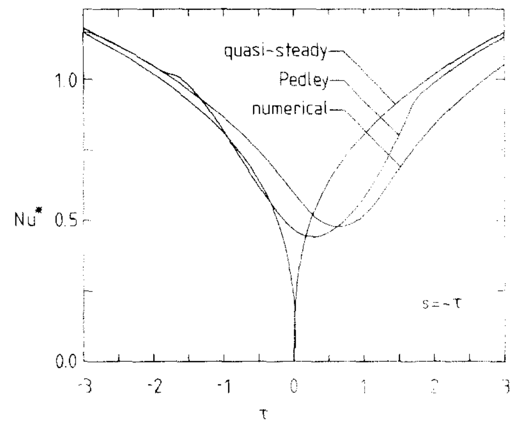

FIG. 7. Heat transfer for decelerated flow in comparison with Pedley's results [11].

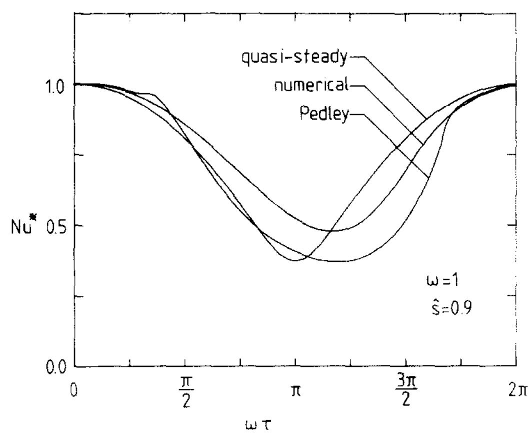

FIG. 5(b). Comparison with Pedley's solution [11] in non-reversing pulsatile flow ($\hat{s} = 0.9$, $\omega = 1.0$).

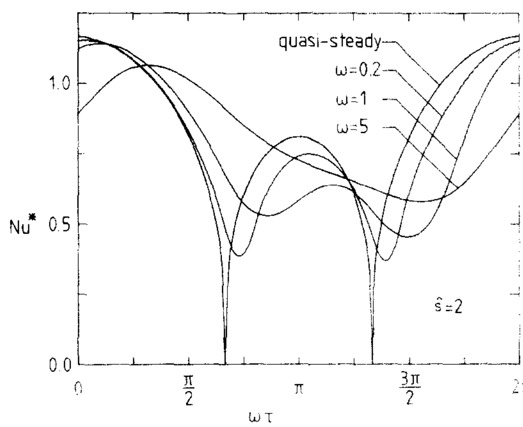

FIG. 8(a). Response of the hot film in pulsatile reversing flow, $\hat{s} = 2.0$.

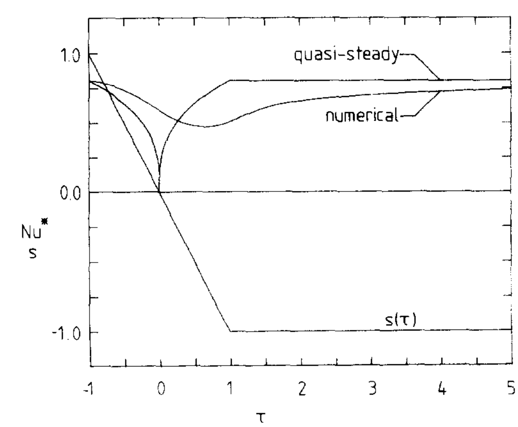

FIG. 6. Heat transfer for the decelerated and constant shear.

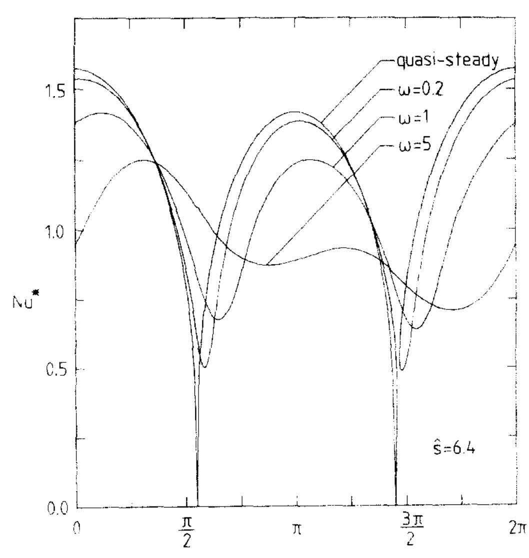

FIG. 8(b). Response of the hot film in pulsatile reversing flow, $\hat{s} = 6.4$.

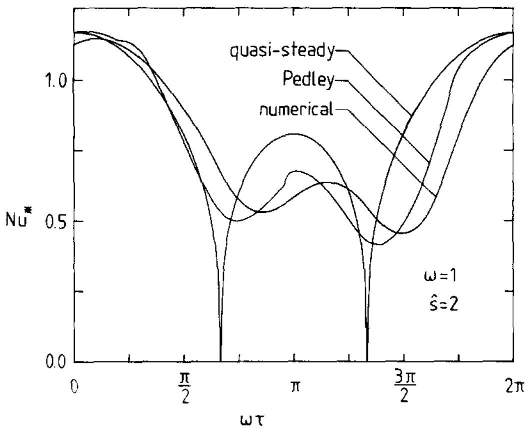

FIG. 9(a). Comparison with Pedley's solution [11] in reversing pulsatile flow, $\hat{s} = 2.0$.

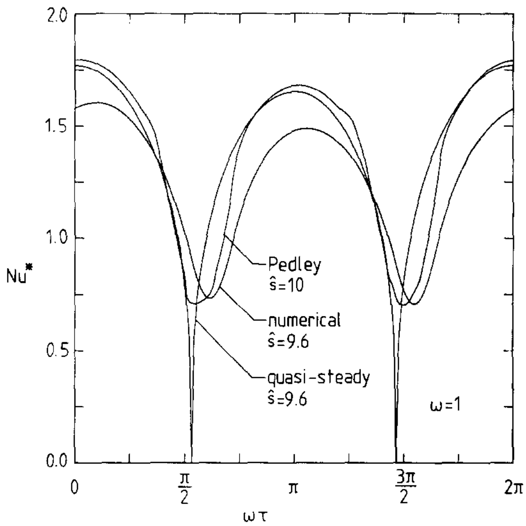

FIG. 9(b). Comparison with Pedley's solution [11] in reversing pulsatile flow, $\hat{s} = 9.6$.

5.1. *Non-reversing shear flow*

Figures 4(a) and (b) show the results of the numerical heat transfer calculations in non-reversing flow for different values of the amplitude $\hat{s}$ and the frequency parameter ω. The pulsations in heat transfer due to the unsteady flow field are quite pronounced even for high ω. But the minimum as well as the maximum value of heat transfer are attenuated and have a phase lag in comparison with the quasi-steady heat flux. The attenuation and the phase lag are strongest for high frequencies and high amplitudes of the pulsation, as can be expected. The attenuation can be expressed by an amplitude ratio

$$A_{R\max} = \frac{Nu^*_{\max} - Nu^*_s}{Nu^*_{q\max} - Nu^*_s} \tag{24}$$

and the phase shift is given by

$$\phi_{\max} = \omega\tau(Nu^*_{\max}) - \omega\tau(Nu^*_{q\max}) \tag{25}$$

(with similar formulae for the minimum).

Graphs of the variations of A_R and ϕ with the parameters $\hat{s}$ and ω are shown in Figs. 10(a), (b), 11(a) and (b). The distortion of the quasi-steady heat transfer curve by thermal inertia effects is more discernible at the minimum where convection is small and thermal inertia is important than at the maximum where convection dominates and the influence of thermal inertia is comparatively small. These differences between the maxima and minima cannot be detected by a first order perturbation solution such as Lighthill's theory [3].

For pulsatile flow with the amplitude $\hat{s} = 0.5$ [Fig. 4(a)] departures from quasi-steady behaviour are small as long as ω remains smaller than about 1.25. Then the amplitude response A_R is accurate to 10% and the maximum phase lag is smaller than 22°. With

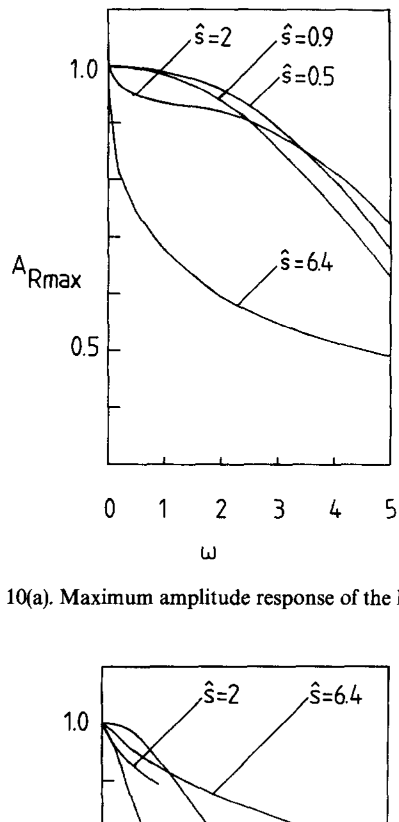

FIG. 10(a). Maximum amplitude response of the hot film.

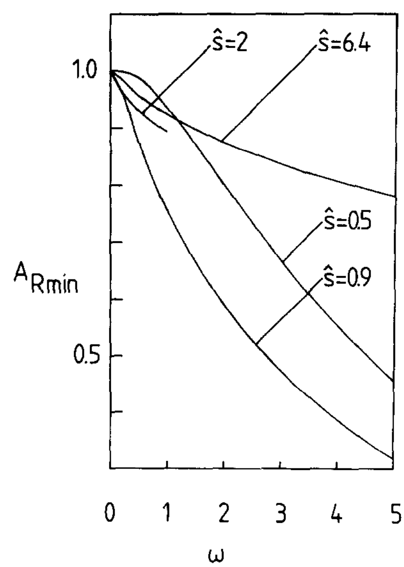

FIG. 10(b). Minimum amplitude response of the hot film.

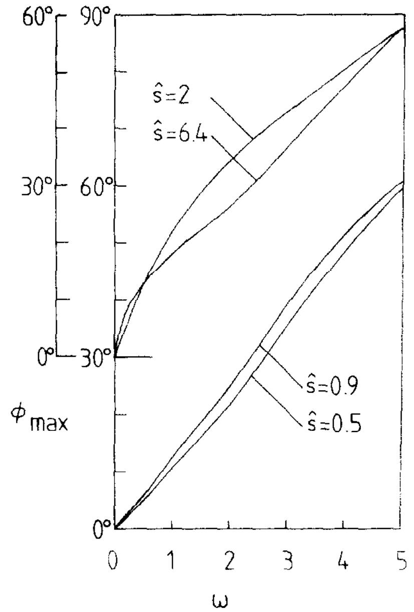

FIG. 11(a). Maximum phase response of the hot film.

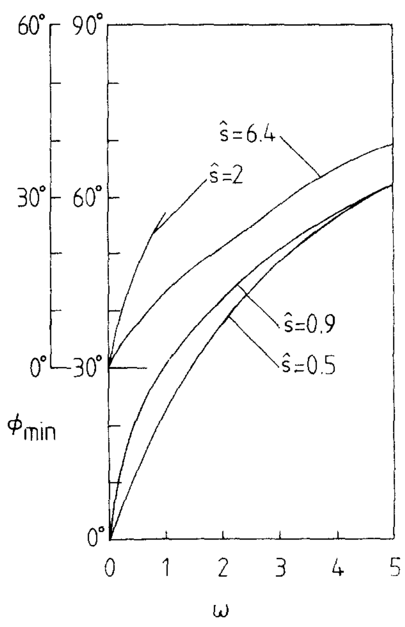

FIG. 11(b). Minimum phase response of the hot film.

increasing ω the departures from quasi-steady behaviour grow rapidly. Similarly, an increase in amplitude $\hat{s}$ soon leads to a deterioration of the hot-film signal. So, for example, the amplitude response for $\hat{s} = 0.9$ [Fig. 4(b)] is 10% accurate only as long as the frequency parameter does not exceed 0.4.

In Figs. 5(a) and (b) the numerical results are compared with the heat transfer calculations of Pedley [11]. His theory gives very accurate results for high shear rates (Pedley's "approximately quasi-steady regime") although the slight phase lag cannot be predicted. In the range of low shear rates Pedley assumes a "purely diffusive regime" and neglects the convection term, solving only the 1-dim. heat conduction equation. This assumption, however leads to an underestimate in heat transfer at times of low fluid velocities. (Pedley matched his two solutions by making the "centre of mass" of the temperature profile continuous during the change: "approximately quasi-steady" "purely diffusive". A matching for the changeover "purely diffusive"–"approximately quasi-steady" could not be made as the "approximately quasi-steady" solution is independent of initial temperatures. Thus, in reversing flow, the influence of the thermal wake had to be neglected.)

5.2. *Reversing shear flow*

In reversing shear flow the numerical calculations show a considerable influence of the thermal wake on heat transfer. For the decelerated and constant shear (Fig. 6) the backflow of the wake (i.e. the increase in ambient fluid temperature) reduces the Nusselt number significantly. After 4 time units τ of constant shear the influence of thermal inertia due to previous changes in the fluid velocity will be damped out but still the heat transfer is reduced by 8%. This must be due to the thermal wake which remains at a fairly high temperature for a long distance [as can also be seen from the steady wake solution, equation (6)].

In the case of uniformly decelerated shear flow [equation (13)] the same effect can be seen, however, combined with thermal inertia effects (Fig. 7). For comparison, Pedley's approximate solution is plotted in the same figure as well as the quasi-steady approximation. The different influences on heat transfer can easily be explained by comparing these curves. For positive shear ($\tau < -1.5$) Nu^* is higher than the quasi-steady heat flux Nu_q^* owing to the decelerating influence of the thermal inertia. This is reflected by both the numerical and the approximate solution and their agreement is good. For low shear rates (positive and negative) the curves do not follow the quasi-steady one and remain positive. The neglect of convection in Pedley's theory, however, leads to an underestimate in heat transfer and there is a time lag between the approximate and the numerical solution. For high negative shear ($\tau > 1.5$) Pedley's solution cannot detect the changes introduced by the backflow of the thermal wake but it accounts for the inertial effects. Thus, the difference between the quasi-steady and the approximate solution reflects the influence of thermal inertia whereas the further difference to the numerical curve shows the influence of the thermal wake. The results for pulsatile flow with shear reversal are presented in Figs. 8(a), (b), 9(a) and (b). For the amplitude $\hat{s} = 2$ [Fig. 8(a)] the maximum value of heat transfer is well represented and the amplitude response for the maximum is good although there is a phase lag between heat transfer and shear variation. Only for high frequencies ($\omega \gtrsim 2$) the response of the maximum is down by more than 10%. The minimum response in reversing flow has to be redefined because the hot-film signal does not show the change of the flow direction (change of sign):

$$A_{R\min} = \frac{Nu^*_{\min} + Nu^*_s}{Nu^*_{q\min} + Nu^*_s}. \quad (26)$$

This amplitude response is in general better in reversing than in non-reversing flow because the hot-film signal remains positive and the thermal inertia keeps the heat flux high, reducing the decrease in heat transfer due to the backflow of the thermal wake. Thus, the relative error in the amplitude response remains small (also because of the high amplitude). However, the sensitivity of the hot film to changes in the flow field is bad during the period of reversed shear: the instantaneous value of the wall shear can only be predicted inaccurately from the measured heat transfer, and for high frequencies ($\omega \gtrsim 2$) the hot-film signal for the reversed shear is completely smeared out (see curve for $\omega = 5$) making accurate measurements impossible.

Similar behaviour occurs for $\hat{s} = 6.4$ [Fig. 8(b)] but here the quality of the hot-film response is better than for $\hat{s} = 2$. The phase difference between flow pulse and heat transfer is reduced and the heat transfer for the reversed shear follows the quasi-steady heat flux slightly better. Though these improvements are striking at first glance (one would expect a worse behaviour because of the rapid changes in wall shear) they can easily be explained as follows. The absolute values of the shear function are high most of the time (for positive as well as negative shear); thus, the thermal disturbances (inertia, wake) are small and convection dominates.

The maximum heat transfer, however, is changed by heated fluid which was convected in front of the film during backflow and now passes the film again reducing its heat output. This does not happen for $\hat{s} = 2$ where at times of maximum shear all heated fluid is already washed away. Therefore the hot-film signal quality is, in the neighbourhood of the maximum shear rate, worse for high amplitudes than for low amplitudes.

Computations were made only up to the third flow reversal. The curves shown represent the last period (from the beginning of the second up to the beginning of the third backflow period) of these calculations. In order to present the plots with the argument of the flow pulse ($\omega\tau = 0$ to $\omega\tau = 2\pi$) the last part of the curve was cut off and joined at the front (before the second shear reversal). The small kink in the very unsteady curve $\hat{s} = 6.4/\omega = 5$ at the joining point shows that the heat transfer oscillations were not quite periodic at that time whereas the smoothness of the other curves indicate the fact that all initial transients were completely damped out.

Pedley's [11] approximate solution gives similar wave forms as the numerical results presented here [Figs. 9(a) and (b)] but these have a significant phase advance. A phase difference with quantitatively good agreement was also found by Pedley in a comparison of his results with experimental data [13, 14] and the most probable cause was thought to originate from the three-dimensionality of the flow and temperature field. Although the present work cannot explain the phase lag of Pedley's solution completely, and 3-dim. effects must still be considered important (the phase lag between Pedley's solution and experiments is about $\pi/4$; the one between his solution and the numerical results is at most $\pi/8$, however) there seem to be inaccurate predictions with Pedley's theory because of its strong simplifications. The good quantitative agreement might occur because two counteractive effects (the convection at low shear rates and backflow of the wake) were omitted which will, at least partially, cancel. Thus, their neglect does not introduce large errors at times of low shear. In the range of higher shear the neglect of the increase in ambient fluid temperature due to the thermal wake leads to an overestimate of heat transfer in Pedley's solution.

Figures 10(a), (b) and 11(a), (b) show the curves of the characteristic quantities of a hot-film signal, the amplitude and the phase response, for different amplitudes as a function of the frequency parameter ω. They were plotted from the calculated data for $\omega = 0.2$, 0.5, 1, 2, 3.5 and 5. The exact location and the values of the maxima and minima were calculated from the discrete numerical points [9° (non-reversing) and about 2° (reversing) apart] by a cubic spline interpolation for the five points next to the extrema.

5.3. *Time-averaged heat transfer*

The question of whether a superimposed oscillation on a steady stream enhances heat or mass transfer when the average flow rate is held constant will be discussed briefly as it is related to this investigation. The time-averaged heat transfer

$$\overline{Nu^*} = \frac{1}{2\pi} \int_0^{2\pi} Nu^* \, d(\omega\tau) \quad (27)$$

was evaluated by a Simpson rule integration of the instantaneous heat transfer values. In flow without reversal [Fig. 12(a)] the time averaged heat transfer is always decreased as could also be shown by McMichael and Hellums [23]. The reduction in heat transfer is the larger the smaller the frequency and the higher the amplitude of the pulsation. For high frequencies the calculated averaged heat transfer approaches the steady value Nu^*_s.

Calculations of the time averaged heat transfer in reversing pulsatile flow exhibit a different behaviour which is strongly dependent on amplitude [Fig. 12(b)]. If the backflow remains small the ratio $\overline{Nu^*}/Nu^*_s$ remains smaller than one. For increasing frequency it first drops from the quasi-steady value owing to the influence of the wake but increases then again owing to thermal inertia to approach finally the value one.

For high amplitudes there is an increase in time averaged heat transfer but the effect of the thermal wake as well as the thermal inertia counteract the improvement of heat flux such that best values are obtained for low frequencies.

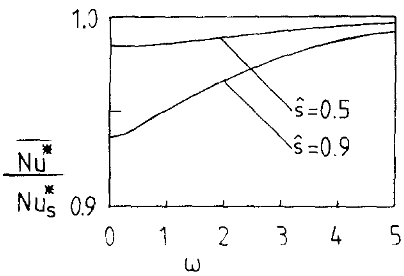

FIG. 12(a). Time averaged heat flux in non-reversing pulsating flow.

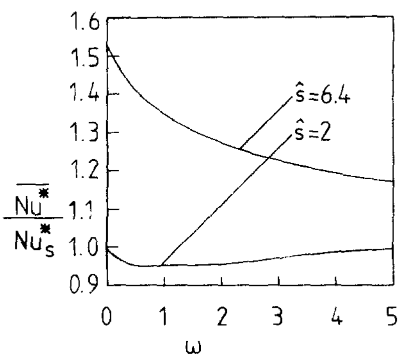

FIG. 12(b). Time averaged heat flux in reversing pulsating flow.

The following general results can be obtained from Figs. 12(a) and (b): an enhancement of time averaged heat transfer is only possible in flow with reversal and can only be attained if the quasi-steady heat flux (for the same flow field) also shows an enhancement. The improvement of heat transfer in flow that fulfils the conditions is the better the smaller the pulse frequency and the higher the amplitude. For very high frequencies the ratio $\overline{Nu^*}/Nu_s$ approaches one in all cases.

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CONVECTION THERMIQUE FORCEE ET VARIABLE AUTOUR D'UN FILM CHAUD DANS UN ECOULEMENT CISAILLANT AVEC OU SANS RETOUR

Résumé—Le flux thermique à partir d'un film chaud à température constante, sur une surface plane isolée, est étudié numériquement en supposant que la théorie de couche limite thermique est applicable partour et à tout instant. Le champ d' écoulement est décrit par (a) une pulsation ou (b) une décélération linéaire d'un écoulement cisaillant bidimensionnel. Pour les grandes fréquences de pulsation le transfert thermique calculé s'écarte fortement de la valeur quasi-permanente. Dans l'écoulement de retour, on peut constater une influence considérable du sillage thermique. Des calculs du flux thermique moyen dans le temps montrent qu'un accroissement du transfert dû à la superposition des oscillations peut se produire seulement en écoulement de retour.

INSTATIONÄRER WÄRMEÜBERGANG DURCH ERZWUNGENE KONVEKTION VON EINEM HEISSFILM IN SCHERSTRÖMUNGEN OHNE UND MIT RICHTUNGSUMKEHR

Zusammenfassung—Die Wärmeabgabe eines Konstant-Temperatur-Heißfilms, der glatt in eine ebene, isolierende Oberfläche eingebaut ist, wird numerisch unter der Annahme untersucht, daß die thermischen Grenzschichtvereinfachungen überall und zu allen Zeiten gültig sind. Das Strömungsfeld wird durch (a) eine pulsierende oder (b) eine gleichmäßig verzögerte, ebene Scherströmung beschrieben. Der berechnete, instationäre Wärmeübergang weicht für hohe Pulsationsfrequenzen stark vom quasistationären Wert ab. In Strömungen mit Richtungsumkher wird ein erheblicher Einfluß des thermischen Nachlaufs deutlich. Berechnungen des zeitlich gemittelten Wärmeübergangs zeigen, daß nur in Strömungen mit Richtungswechsel eine Verbesserung der Wärmeabgabe durch aufgeprägte Oszillationen möglich ist.

НЕСТАЦИОНАРНЫЙ ТЕПЛОПЕРЕНОС ВЫНУЖДЕННОЙ КОНВЕКЦИЕЙ ОТ НАГРЕТОЙ ПЛЕНКИ ПРИ ПРЯМОМ И ОБРАТНОМ ТЕЧЕНИИ ВЯЗКОЙ ЖИДКОСТИ

Аннотация — Проведено численное исследование переноса тепла от нагретой пленки постоянной температуры, расположенной на поверхности плоской изолирующей стенки, в предположении, что теория теплового пограничного слоя применима в любой точке и в любой момент времени. Поле течения описывается с помощью (а) пульсирующего или (б) линейно замедляющегося потока с поперечным градиентом скорости. При высоких частотах пульсаций рассчитанные значения нестационарного теплового потока значительно отличаются от квазистационарных значений. При обратном течении наблюдается существенное влияние теплового следа. Расчеты усредненных по времени величин теплового потока показывают, что интенсификация теплопереноса за счет пульсаций возможна только при обратном течении.